A short proof of an Erdős-Ko-Rado theorem for compositions

Peter Borg

Department of Mathematics, University of Malta, Malta
p.borg.02@cantab.net

Abstract

If $a_1, \dots, a_k$ and n are positive integers such that $n = a_1 + \dots + a_k$, then the tuple $(a_1, \dots, a_k)$ is a *composition of* n of *length* k. We say that $(a_1, \dots, a_k)$ *strongly t-intersects* $(b_1, \dots, b_k)$ if there are at least t distinct indices i such that $a_i = b_i$. A set A of compositions is *strongly t-intersecting* if every two members of A strongly t-intersect. Let $C_{n,k}$ be the set of all compositions of n of length k. Ku and Wong [An analogue of the Erdős-Ko-Rado theorem for weak compositions, Discrete Mathematics 313 (2013), 2463–2468] showed that for every two positive integers k and t with $k \geq t+2$, there exists an integer $n_0(k,t)$ such that for $n \geq n_0(k,t)$, the size of any strongly t-intersecting subset A of $C_{n,k}$ is at most $\binom{n-t-1}{n-k}$, and the bound is attained if and only if $A = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_{i_1} = \dots = a_{i_t} = 1\}$ for some distinct $i_1, \dots, i_t$ in $\{1, \dots, k\}$. We provide a short proof of this analogue of the Erdős-Ko-Rado Theorem. It yields an improved value of $n_0(k,t)$. We also show that the condition $n \geq n_0(k,t)$ cannot be replaced by $k \geq k_0(t)$ or $n \geq n_0(t)$ (that is, the dependence of n on k is inevitable), and we suggest a Frankl-type conjecture about the extremal structures for any n, k and t.

1 Introduction

Recently, Ku and Wong [18] proved an analogue of the classical Erdős-Ko-Rado Theorem [7] for weak compositions. In this note we provide a short proof of their result. We set up the main definitions and notation before stating the result.

Unless otherwise stated, we will use small letters such as x to denote non-negative integers or functions or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose members are sets themselves). We call a set A an *r-element set* if its size $|A|$ is r. The family of all subsets of a set X is denoted by 2^X, and the family of all r-element subsets of X is denoted by $\binom{X}{r}$. For any integer $n \geq 1$, the set $\{1, \dots, n\}$ of the first n positive integers is denoted by $[n]$.

If $a_1, \dots, a_k$ and n are positive integers such that $n = \sum_{i=1}^k a_i$, then the k-tuple $(a_1, \dots, a_k)$ is a *composition of* n of *length* k. If $a_1, \dots, a_k$ and n are non-negative integers such that $n = \sum_{i=1}^k a_i$, then $(a_1, \dots, a_k)$ is a *weak composition of* n of *length* k. Let $C_{n,k}$ be the set of all compositions of n of length k, and let $W_{n,k}$ be the set of all weak compositions of n of length k. An elementary counting result is that $|W_{n,k}| = \binom{n+k-1}{n}$. Since $W_{n-k,k} = \{(a_1 - 1, \dots, a_k - 1) \colon (a_1, \dots, a_k) \in C_{n,k}\}$, $|C_{n,k}| = \binom{n-1}{n-k}$.

We say that $(a_1, \dots, a_k)$ *strongly t-intersects* $(b_1, \dots, b_k)$ if there exists $T \in \binom{[k]}{t}$ such that $a_i = b_i$ for each $i \in T$. We call a set A of k-tuples *strongly t-intersecting* if every two members of A strongly t-intersect.

Recently, Ku and Wong [18] proved the following result.

Theorem 1.1 ([18]) *For every two positive integers k and t with $k \geq t+2$, there exists an integer $n_0(k,t)$ such that for $n \geq n_0(k,t)$, the size of every strongly t-intersecting subset A of $W_{n,k}$ is at most $\binom{n+k-t-1}{n}$, and the bound is attained if and only if for some $T \in \binom{[k]}{t}$, $A = \{(a_1, \dots, a_k) \in W_{n,k} \colon a_i = 0$ for each $i \in T\}$.*

We provide a short proof of this result. It yields an improved value of $n_0(k,t)$. Let $c(k,t) = (k-t-1)\binom{3k-2t-1}{t+1} + t + 2$.

Theorem 1.2 *If $t \geq 1$, $k \geq t+2$, $n \geq c(k,t)$, and A is a strongly t-intersecting subset of $C_{n,k}$, then*

$$|A| \leq \binom{n-t-1}{n-k},$$

and equality holds if and only if for some $T \in \binom{[k]}{t}$, $A = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_i = 1$ for each $i \in T\}$.

This gives Theorem 1.1 as follows. Let $t \geq 1$, $k \geq t+2$, and $n \geq c(k,t) - k$. Let A be a strongly t-intersecting subset of $W_{n,k}$, and let $A' = \{(a_1+1, \dots, a_k+1) \colon (a_1, \dots, a_k) \in A\}$. So A' is a strongly t-intersecting subset of $C_{n',k}$, where $n' = n+k \geq c(k,t)$. By Theorem 1.2, $|A'| \leq \binom{n'-t-1}{n'-k} = \binom{n+k-t-1}{n}$, and equality holds if and only if for some $T \in \binom{[k]}{t}$, $A' = \{(a'_1, \dots, a'_k) \in C_{n',k} \colon a'_i = 1$ for each $i \in T\}$. So $|A| \leq \binom{n+k-t-1}{n}$, and equality holds if and only if for some $T \in \binom{[k]}{t}$, $A = \{(a_1, \dots, a_k) \in W_{n,k} \colon a_i = 0$ for each $i \in T\}$. By a similar argument, Theorem 1.1 implies Theorem 1.2 for $n \geq n_0(k,t) + k$.

The problem is trivial for $t \leq k \leq t+1$. Let A be a strongly t-intersecting subset of $C_{n,k}$. If $k = t$, then A can only have one element. If $k = t+1$ and $(a_1, \dots, a_k), (b_1, \dots, b_k) \in A$, then for some $T \in \binom{[k]}{t}$, we have $a_i = b_i$ for $i \in T$. Only one index is outside T. Since $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ have the same sum n, they must therefore also agree in the remaining position, so $|A| = 1$.

The value of $n_0(k,t)$ obtained in [18] for Theorem 1.1 is $\max\{\left((k-t-1)\binom{k}{t}\right)^2, (2k-2t)^{2^{k-t+1}} + 1\}$. As we pointed out above, Theorem 1.1 holds with $n \geq c(k,t) - k$. It follows that Theorem 1.1 holds with $n \geq (k-t-1)(3k-2t-1)^{t+1}$.

The dependence of n on t in Theorem 1.2 can be avoided by taking n to be sufficiently large. A crude way of showing this is that $c(k,t) \leq k\binom{3k-2t-1}{t+1} < k\binom{3k}{t+1} < k\binom{3k}{\lfloor 3k/2 \rfloor}$; so the result is true for $n \geq k\binom{3k}{\lfloor 3k/2 \rfloor}$. In Section 3 we show that the dependence of n on k is inevitable and that we cannot even replace $n \geq c(k,t)$ by $k \geq k_0(t)$.

Theorems 1.1 and 1.2 are analogues of the classical Erdős-Ko-Rado (EKR) Theorem [7], which inspired many results in extremal set theory (see [6, 10, 8, 3]). A family $\mathcal{A}$ of sets is *t-intersecting* if every two sets in $\mathcal{A}$ have at least t common elements. The EKR Theorem says that for $1 \leq t \leq k$, there exists an integer $n_0(k,t)$ such that for $n \geq n_0(k,t)$, the size of any t-intersecting subfamily of $\binom{[n]}{k}$ is at most $\binom{n-t}{k-t}$, which is the size of the simplest t-intersecting subfamily $\{A \in \binom{[n]}{k} \colon [t] \subseteq A\}$. It was also shown in [7] that the smallest possible value of $n_0(k,1)$ is $2k$. There are various proofs of this (see [16, 11, 14, 5]), two of which are particularly short and beautiful: Katona's [14], introducing the elegant cycle method, and Daykin's [5], using the powerful Kruskal-Katona Theorem [15, 17]. Frankl [9] showed that for $t \geq 15$, the smallest possible value of $n_0(k,t)$ is $(k-t+1)(t+1)$. Subsequently, Wilson [20] proved this for all $t \geq 1$. Frankl [9] conjectured that the size of a largest t-intersecting subfamily of $\binom{[n]}{k}$ is $\max\{|\{A \in \binom{[n]}{k} \colon |A \cap [t+2i]| \geq t+i\}| \colon i \in \{0\} \cup [k-t]\}$. A remarkable proof of this conjecture together with the complete characterisation of the extremal structures was obtained by Ahlswede and Khachatrian [1]. The t-intersection problem for $2^{[n]}$ was completely solved by Katona [16]. These are prominent results in extremal set theory.

As will become clearer in the proof, Theorem 1.2 can also be phrased in terms of t-intersecting subfamilies of a family. Indeed, it is equivalent to the following: if $n \geq c(k,t)$ and $\mathcal{A}$ is a t-intersecting subfamily of the family $\mathcal{C}_{n,k} = \{\{(1,a_1),\dots,(k,a_k)\} \colon (a_1,\dots,a_k) \in C_{n,k}\}$, then $|\mathcal{A}| \leq \binom{n-t-1}{n-k}$, and equality holds if and only if for some $T \in \binom{[k]}{t}$, $\mathcal{A} = \{\{(1,a_1),\dots,(k,a_k)\} \in \mathcal{C}_{n,k} \colon a_i = 1 \text{ for each } i \in T\}$.

EKR-type results have been obtained in a wide variety of contexts, many of which are surveyed in [6, 10, 8, 3]. Usually the objects have symmetry properties (see [4, Section 3.2] and [19]) or enable use of *compression* operators (also called *shift* operators) to push t-intersecting families towards a desired form (see [10, 13, 12]). One of the main motivating factors behind this note is that although the family $\mathcal{C}_{n,k}$ does not have any of these properties, we can still determine its largest t-intersecting subfamilies for n sufficiently large, using more than one method. It is interesting that Ku and Wong [18] managed to take an inductive approach. We will show that the method in [7] can be adapted to this framework. However, since $\mathcal{C}_{n,k}$ does not have any of the above properties, the problem of determining the maximum size of a t-intersecting subfamily of $\mathcal{C}_{n,k}$ for any n, k and t must be very hard. We conjecture that the extremal structures are similar to those in the above-mentioned conjecture of Frankl (proved in [1]). We state the conjecture using the original formulation.

Conjecture 1.3 *Let $1 \leq t \leq k \leq n$. For $i = 0,\dots,\lfloor\frac{k-t}{2}\rfloor$, let $A_i = \{(a_1,\dots,a_k) \in C_{n,k} \colon |\{j \in [t+2i] \colon a_j = 1\}| \geq t+i\}$. The size of a largest strongly t-intersecting subset of $C_{n,k}$ is* $\max\{|A_i| \colon 0 \leq i \leq \lfloor\frac{k-t}{2}\rfloor\}$.

2 Proof of Theorem 1.2

A t-intersecting family is *non-trivial* if its members have fewer than t common elements.

The following lemma emerges from [7] (see also [2, Proof of Theorem 2.1]).

Lemma 2.1 *If $\mathcal{A}$ is a non-trivial t-intersecting family whose members are of size at most k, then there exists a set J of size at most $3k-2t-1$ such that $|A \cap J| \geq t+1$ for each $A \in \mathcal{A}$.*

This lemma is the key ingredient of the proof of Theorem 1.2, which we can now provide. As indicated in Section 1, we transform the setting of compositions to a setting of sets of pairs.

Proof of Theorem 1.2. Let $k \geq t+2$ and $n \geq c(k,t)$. Let A be a non-empty strongly t-intersecting subset of $C_{n,k}$. Write $\mathbf{a}$ for a composition $(a_1,\dots,a_k)$. Let $S_{\mathbf{a}} = \{(i,a_i) \colon i \in [k]\}$. Let $\mathcal{C}_{n,k} = \{S_{\mathbf{a}} \colon \mathbf{a} \in C_{n,k}\}$. Let $f : C_{n,k} \to \mathcal{C}_{n,k}$ such that $f(\mathbf{a}) = S_{\mathbf{a}}$ for each $\mathbf{a} \in C_{n,k}$. Clearly, f is a bijection. Note that two compositions $\mathbf{a}$ and $\mathbf{b}$ strongly t-intersect if and only if $|S_{\mathbf{a}} \cap S_{\mathbf{b}}| \geq t$. Thus, a subset I of $C_{n,k}$ is strongly t-intersecting if and only if $\{S_{\mathbf{a}} \colon \mathbf{a} \in I\}$ is a t-intersecting subfamily of $\mathcal{C}_{n,k}$.

Letting $\mathcal{A} = \{f(\mathbf{a}) \colon \mathbf{a} \in A\}$, we have that $|\mathcal{A}| = |A|$, $\mathcal{A}$ is a t-intersecting subfamily of $\mathcal{C}_{n,k}$, and $|X| = k$ for each $X \in \mathcal{A}$.

Suppose that the sets in $\mathcal{A}$ have t common elements $(h_1, d_{h_1}), \dots, (h_t, d_{h_t})$. Let $D = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_{h_i} = d_{h_i}$ for each $i \in [t]\}$. Thus $A \subseteq D$. Let $p = \sum_{i=1}^{t} d_{h_i}$. Note that $|D| = |C_{n-p,k-t}| = \binom{n-p-1}{n-p-k+t}$. Since $d_{h_i} \geq 1$ for each $i \in [t]$, we have $p \geq t$. Hence $|D| \leq \binom{n-t-1}{n-k}$, and equality holds if and only if $p = t$. Now $p = t$ if and only if $d_{h_i} = 1$ for each $i \in [t]$. Thus $|A| \leq \binom{n-t-1}{n-k}$, and equality holds if and only if $A = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_{h_i} = 1$ for each $i \in [t]\}$.

Now suppose that the sets in $\mathcal{A}$ do not have t common elements, so $\mathcal{A}$ is a non-trivial t-intersecting family. By Lemma 2.1, there exists a set J such that $|J| \leq 3k - 2t - 1$ and $|X \cap J| \geq t + 1$ for each $X \in \mathcal{A}$. Thus $\mathcal{A} \subseteq \bigcup_{T \in \binom{J}{t+1}} \{X \in \mathcal{C}_{n,k} \colon T \subset X\}$. Let $T^* \in \binom{J}{t+1}$ such that $|\{X \in \mathcal{C}_{n,k} \colon T \subset X\}| \leq |\{X \in \mathcal{C}_{n,k} \colon T^* \subset X\}|$ for all $T \in \binom{J}{t+1}$. Let $\mathcal{B} = \{X \in \mathcal{C}_{n,k} \colon T^* \subset X\}$. We have

$$\begin{aligned} |\mathcal{A}| &\leq \left| \bigcup_{T \in \binom{J}{t+1}} \{X \in \mathcal{C}_{n,k} \colon T \subset X\} \right| \leq \sum_{T \in \binom{J}{t+1}} |\{X \in \mathcal{C}_{n,k} \colon T \subset X\}| \leq \sum_{T \in \binom{J}{t+1}} |\mathcal{B}| \\ &= \binom{|J|}{t+1} |\mathcal{B}| \leq \binom{3k-2t-1}{t+1} |\mathcal{B}|. \end{aligned}$$

Let $B = \{f^{-1}(X) \colon X \in \mathcal{B}\}$, so $|B| = |\mathcal{B}|$. Let $(l_1, e_{l_1}), \dots, (l_{t+1}, e_{l_{t+1}})$ be the elements of T^*. Now $B = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_{l_i} = e_{l_i}$ for each $i \in [t+1]\}$. Let $q = \sum_{i=1}^{t+1} e_{l_i}$. We have $q \geq t + 1$ and

$$|B| = |C_{n-q,k-(t+1)}| = \binom{n-q-1}{n-q-k+(t+1)} \leq \binom{n-(t+1)-1}{n-(t+1)-k+(t+1)} = \binom{n-t-2}{n-k}.$$

Hence $|A| = |\mathcal{A}| \leq \binom{3k-2t-1}{t+1} |\mathcal{B}| \leq \binom{3k-2t-1}{t+1} \binom{n-t-2}{n-k}$. Now $\binom{n-t-1}{n-k} = \frac{n-t-1}{k-t-1} \binom{n-t-2}{n-k}$. Thus, since $n \geq c(k,t)$, we have $\binom{n-t-1}{n-k} > \binom{3k-2t-1}{t+1} \binom{n-t-2}{n-k}$, and $|A| < \binom{n-t-1}{n-k}$. □

3 Dependence on k

We now show that the dependence of n on k is inevitable. Let $T_{n,k} = \{(a_1, \dots, a_k) \in C_{n,k} \colon a_i = 1$ for each $i \in [t]\}$ and $N_{n,k} = \{(a_1, \dots, a_k) \in C_{n,k} \colon |\{i \in [t+2] \colon a_i = 1\}| \geq t + 1\}$. Note that $T_{n,k}$ and $N_{n,k}$ are strongly t-intersecting subsets of $C_{n,k}$, and $T_{n,k}$ is one of the optimal families given by Theorem 1.2. We have

$$\begin{aligned} |N_{n,k}| - |T_{n,k}| &= \big((t+2)|C_{n-t-1,k-t-1}| - (t+1)|C_{n-t-2,k-t-2}|\big) - |C_{n-t,k-t}| \\ &= (t+2)\binom{n-t-2}{n-k} - (t+1)\binom{n-t-3}{n-k} - \binom{n-t-1}{n-k} \\ &= \binom{n-t-2}{n-k} \left(t + 2 - (t+1)\frac{k-t-2}{n-t-2} - \frac{n-t-1}{k-t-1} \right) \\ &= \binom{n-t-2}{n-k} \left(\frac{(t+1)(n-k)}{n-t-2} - \frac{n-k}{k-t-1} \right) \\ &= (n-k)\binom{n-t-2}{n-k} \left(\frac{(t+1)(k-t)+1-n}{(n-t-2)(k-t-1)} \right). \end{aligned}$$

Thus, $|N_{n,k}| > |T_{n,k}|$ if $k + 1 \leq n \leq (t+1)(k-t)$. No matter how large k or n is, Theorem 1.2 does not hold if $k+1 \leq n \leq (t+1)(k-t)$. In other words, we cannot replace $n \geq c(k,t)$ by $n \geq n_0(t)$ or $k \geq k_0(t)$.

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